Reduced basis for variational inequalities
Application to contact mechanics

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Motivations for contact

- Long-standing question in many contexts...
- Present industrial context
  - Behavior of valve components
  - Costly simulations (≈ 12h using ‘code_aster’)
- Nonlinearities, constraints

Challenge

- Expensive for parameterized studies

Goal

- Nonlinearly constrained model reduction
Variational inequalities with linear constraints

**Reduced basis procedures**

- **Haasdonk, Salomon, Wohlmuth.** A reduced basis method for parametrized variational inequalities. 2012.
- **Balajewicz, Amsallem, Farhat.** Projection-based model reduction for contact problems. 2016.
- **Fauque, Ramière, Ryckelynck.** Hybrid hyper-reduced modeling for contact mechanics problems. 2018.

**Related work**

- **Burkovska, Haasdonk, Salomon, Wohlmuth.** Reduced basis methods for pricing options with the Black-Scholes and Heston models. 2014.
- **Glas, Urban.** Numerical investigations of an error bound for reduced basis approximations of noncoercive variational inequalities. 2015.
- **Bader, Zhang, Veroy.** An empirical interpolation approach to reduced basis approximations for variational inequalities, 2016.
Outline

1 Elastic contact problem
2 Abstract model problem
3 The reduced-basis model
Elastic contact problem

Strain tensor

$$\varepsilon(v) := \frac{1}{2}(\nabla v + \nabla v^T)$$

Stress tensor

$$\sigma(v) = \frac{E\nu}{(1 + \nu)(1 + 2\nu)}\text{tr}(\varepsilon(v))I + \frac{E}{(1 + \nu)}\varepsilon(v)$$

$E$: Young modulus  \quad $\nu$: Poisson coefficient

Parametric equilibrium condition

$$\nabla \cdot \sigma(u(\mu)) = \ell(\mu) \quad \text{in} \quad \Omega(\mu)$$

$$a(\mu; v, w) = \int_{\Omega(\mu)} \sigma(v) : \varepsilon(w) \quad \text{and} \quad f(\mu; w) = \int_{\Omega(\mu)} \ell(\mu)w$$

Other nonlinearities can be handled
Non-interpenetration condition

**Initial configuration**

- $\Gamma_1^c(\mu)$
- $\tilde{n}_2(\mu, u(\mu))$
- $\rho(\mu, u(\mu))$

**Deformed configuration**

- $\Gamma_2^c(\mu)$

Proof: Benaceur, PhD, 2018.

Amin Benaceur  A RBM for variational inequalities  4/29
Consider \( \mathcal{V}(\mu) = H^1(\Omega_1(\mu)) \times H^1(\Omega_2(\mu)) \)

Admissible solutions are denoted \( u(\mu) = (u_1(\mu), u_2(\mu)) \in \mathcal{V}(\mu) \)

For all \( z \in \Gamma_c^c(\mu) \)

\[
\left( u_1(\mu)(z) - (u_2(\mu) \circ \rho(\mu, u(\mu))) (z) \right) \cdot \tilde{n}_2(\mu, u(\mu))(z) \geq \left( \rho(\mu, u(\mu))(z) - z \right) \cdot \tilde{n}_2(\mu, u(\mu))(z) 
\]

\( \Longrightarrow \) Proof: Benaceur, PhD, 2018.
Non-interpenetration condition

Consider \( \mathcal{V}(\mu) = H^1(\Omega_1(\mu)) \times H^1(\Omega_2(\mu)) \)
Admissible solutions are denoted \( u(\mu) = (u_1(\mu), u_2(\mu)) \in \mathcal{V}(\mu) \)

For all \( z \in \Gamma_1^c(\mu) \)
\[
\begin{aligned}
(u_1(\mu)(z) &- (u_2(\mu) \circ \rho(\mu, u(\mu))) \cdot \tilde{n}_2(\mu, u(\mu))(z)) \\
- k(\mu, u(\mu), u(\mu)) &\geq (\rho(\mu, u(\mu))(z) - z) \cdot \tilde{n}_2(\mu, u(\mu))(z) \\
- g(\mu, u(\mu)) &
\end{aligned}
\]

\[\implies \text{Proof: Benaceur, PhD, 2018.}\]
Elastic contact problem
Abstract model problem
The reduced-basis model

Model problem

\( \Omega(\mu) \): bounded domain in \( \mathbb{R}^d \) with a contact boundary \( \Gamma_{\Omega}^c(\mu) \subset \partial \Omega(\mu) \)

\( \mathcal{V}(\mu) \): Hilbert space on \( \Omega(\mu) \)

\( \mathcal{P} \): parameter set

For many values \( \mu \in \mathcal{P} \): Find \( u(\mu) \in \mathcal{V} \) such that

\[
\begin{align*}
u(\mu) &= \arg\min_{v \in \mathcal{V}(\mu)} \frac{1}{2} a(\mu; v, v) - f(\mu; v) \\
k(\mu, u(\mu); u(\mu)) &\leq g(\mu, u(\mu)) \quad \text{a.e. on } \Gamma_{\Omega}^c(\mu)
\end{align*}
\]

\( k(\mu, \cdot; \cdot) \) is semi-linear: natural for contact problems
handy for iterative solution methods
Lagrangian formulation

We consider

- the convex cone \( \mathcal{W}(\mu) := L^2(\Gamma^c(\mu), \mathbb{R}_+) \)
- the Lagrangian \( \mathcal{L}(\mu) : \mathcal{V}(\mu) \times \mathcal{W}(\mu) \rightarrow \mathbb{R} \) defined as

\[
\mathcal{L}(\mu)(v, \eta) := \frac{1}{2} a(\mu; v, v) - f(\mu; v) + \left( \int_{\Gamma^c(\mu)} k(\mu, v; v) \eta - \int_{\Gamma^c(\mu)} g(\mu, v) \eta \right)
\]

Find \((u(\mu), \lambda(\mu)) \in \mathcal{V}(\mu) \times \mathcal{W}(\mu)\) such that

\[
(u(\mu), \lambda(\mu)) = \operatorname{arg \ minmax}_{v \in \mathcal{V}(\mu), \eta \in \mathcal{W}(\mu)} \mathcal{L}(\mu)(v, \eta)
\]

\(u(\mu)\) is the primal solution, \(\lambda(\mu)\) is the dual solution
Discrete FEM formulation

1. FEM space ($\mathcal{N} \gg 1$)
   \[ V_N(\mu) := \text{span}\{\phi_1(\mu), \ldots, \phi_N(\mu)\} \subseteq \mathcal{V}(\mu) \]

2. FEM convex cone ($\mathcal{R} \gg 1$)
   \[ W_\mathcal{R}(\mu) := \text{span}_+\{\psi_1(\mu), \ldots, \psi_\mathcal{R}(\mu)\} \subseteq \mathcal{W}(\mu) \]
Discrete FEM formulation

1. FEM space \((\mathcal{N} \gg 1)\)
   \[ V_{\mathcal{N}}(\mu) := \text{span}\{\phi_1(\mu), \ldots, \phi_\mathcal{N}(\mu)\} \subset V(\mu) \]

2. FEM convex cone \((\mathcal{R} \gg 1)\)
   \[ W_{\mathcal{R}}(\mu) := \text{span}_+\{\psi_1(\mu), \ldots, \psi_\mathcal{R}(\mu)\} \subset W(\mu) \]

**Algebraic FEM formulation**

\[
(u(\mu), \lambda(\mu)) = \arg \min_{v \in \mathbb{R}^{\mathcal{N}}} \max_{\eta \in \mathbb{R}^\mathcal{R}^+} \left\{ \frac{1}{2} v^T A(\mu) v - v^T f(\mu) + \eta^T (K(\mu, v) v - g(\mu, v)) \right\}
\]

\[
A(\mu)_{ij} = a(\mu; \phi_j(\mu), \phi_i(\mu)) \quad K(\mu, w)_{ij} = \int_{\Gamma^c_1(\mu)} k(\mu, w; \phi_j(\mu)) \psi_i(\mu)
\]

Additional nonlinearity caused by spatial discretization
The Kačanov method

- Iterative procedure
- Consists in solving the following problems: For all $k \geq 1$

\[
(u^k(\mu), \lambda^k(\mu)) = \arg \min_{v \in \mathbb{R}^N} \max_{\eta \in \mathbb{R}_+} \frac{1}{2} v^T A(\mu) v - v^T f(\mu) + \eta^T (K(\mu, u^{k-1}(\mu)) v - g(\mu, \lambda^{k-1}(\mu)))
\]

- Stopping criteria

\[
\frac{\| u^k(\mu) - u^{k-1}(\mu) \|_{\mathbb{R}^N}}{\| u^{k-1}(\mu) \|_{\mathbb{R}^N}} \leq \epsilon_{\text{pr}}^{\text{ka}} \quad \frac{\| \lambda^k(\mu) - \lambda^{k-1}(\mu) \|_{\mathbb{R}^N}}{\| \lambda^{k-1}(\mu) \|_{\mathbb{R}^N}} \leq \epsilon_{\text{du}}^{\text{ka}}
\]
The reduced-basis model: Reference configuration

Geometric mapping $h(\mu)$ defined on a reference domain $\tilde{\Omega}$ (with $I = 2$)

$$h(\mu) : \tilde{\Omega} \rightarrow \Omega(\mu)$$

$$x \mapsto \sum_{i=1}^{I} h_i(\mu, x) 1_{\tilde{\Omega}_i}(x)$$
The reduced-basis model: Reference configuration

Geometric mapping $h(\mu)$ defined on a reference domain $\tilde{\Omega}$ (with $I = 2$)

\[ h(\mu) : \tilde{\Omega} \rightarrow \Omega(\mu) \]

\[ x \mapsto \sum_{i=1}^{I} h_i(\mu, x) 1_{\tilde{\Omega}_i}(x) \]

\[ \Omega_i(\mu) = h(\mu)(\tilde{\Omega}_i) \quad \forall i \in \{1, 2\} \]

\[ \Gamma^c_i(\mu) = h(\mu)(\tilde{\Gamma}^c_i) \quad \forall i \in \{1, 2\} \]

**Reference Hilbert space**

\[ \tilde{\mathcal{V}} := H^1(\tilde{\Omega}; \mathbb{R}^d) \]

**Reference convex cone**

\[ \tilde{\mathcal{W}} := L^2(\tilde{\Gamma}^c_1; \mathbb{R}_+) \]

**Parametric Hilbert space**

\[ \mathcal{V}(\mu) = \tilde{\mathcal{V}} \circ h(\mu)^{-1} \]

**Parametric convex cone**

\[ \mathcal{W}(\mu) = \tilde{\mathcal{W}} \circ h(\mu)^{-1}|_{\Gamma^c_1(\mu)} \]
Primal RB subspace

$$\tilde{V}_N \subset \tilde{V}_N \subset \tilde{V}$$

$$\tilde{V}_N = \text{span}\{\tilde{\theta}_1, \ldots, \tilde{\theta}_N\}$$

Dual RB subcone

$$\tilde{W}_R \subset \tilde{W}_R \subset \tilde{W}$$

$$\tilde{W}_R = \text{span}_+\{\tilde{\xi}_1, \ldots, \tilde{\xi}_R\}$$
Reduced basis spaces

**Primal RB subspace**

$$\mathcal{V}_N \subset \mathcal{V}_N \subset \mathcal{V}$$

$$\mathcal{V}_N = \text{span}\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}$$

**Dual RB subcone**

$$\mathcal{W}_R \subset \mathcal{W}_R \subset \mathcal{W}$$

$$\mathcal{W}_R = \text{span}_+\{\bar{\xi}_1, \ldots, \bar{\xi}_R\}$$

**RB approximations**

$$\hat{u}(\mu) = \sum_{n=1}^{N} \hat{u}_n(\mu)\bar{\theta}_n \circ h(\mu)^{-1}$$

$$\hat{\lambda}(\mu) = \sum_{n=1}^{R} \hat{\lambda}_n(\mu)\bar{\xi}_n \circ h(\mu)^{-1}$$
Reduced problem

\[
(\hat{u}(\mu), \hat{\lambda}(\mu)) = \arg \min_{\hat{v} \in \mathbb{R}^N, \hat{\eta} \in \mathbb{R}_+^N} \max \left\{ \frac{1}{2} \hat{v}^T \hat{A}(\mu) \hat{v} - \hat{v}^T \hat{f}(\mu) + \hat{\eta}^T (\hat{K}(\mu, \hat{v}) \hat{v} - \hat{g}(\mu, \hat{v})) \right\}
\]

\[
\hat{A}(\mu) \in \mathbb{R}^{N \times N} \quad \hat{f}(\mu) \in \mathbb{R}^N \quad \hat{K}(\mu, \hat{v}) \in \mathbb{R}^{R \times N} \quad \hat{g}(\mu, \hat{v}) \in \mathbb{R}^R
\]
Reduced problem

\[(\hat{u}(\mu), \hat{\lambda}(\mu)) = \arg \min_{\hat{\nu} \in \mathbb{R}^N, \hat{\eta} \in \mathbb{R}^R} \min_{\hat{\eta}} \max \left\{ \frac{1}{2} \hat{\nu}^T \hat{A}(\mu) \hat{\nu} - \hat{\nu}^T \hat{f}(\mu) + \hat{\eta}^T (\hat{K}(\mu, \hat{\nu}) \hat{\nu} - \hat{g}(\mu, \hat{\nu})) \right\} \]

\[\hat{A}(\mu) \in \mathbb{R}^{N \times N}, \hat{f}(\mu) \in \mathbb{R}^N, \hat{K}(\mu, \hat{\nu}) \in \mathbb{R}^{R \times N}, \hat{g}(\mu, \hat{\nu}) \in \mathbb{R}^R\]

\[\hat{A}(\mu)_{pn} = a(\mu; \hat{\theta}_n \circ h(\mu)^{-1}, \hat{\theta}_p \circ h(\mu)^{-1})\]

\[\hat{K}(\mu, \hat{\nu})_{pn} = \int_{\Gamma_i^c(\mu)} k(\mu, \sum_{i=1}^{N} \hat{\nu}_i \hat{\theta}_i \circ h(\mu)^{-1}; \hat{\theta}_n \circ h(\mu)^{-1}) \xi_p \circ h(\mu)^{-1}\]

Small (dense) matrices but require FEM reconstructions \(\theta_n \circ h(\mu)^{-1}\)...etc

\[\implies \text{Need to separate } (n, p)\text{- and } \mu\text{-dependencies}\]
Functional separation

For the stiffness matrix, we need a decomposition of the form

\[
\left( \hat{A}(\mu) \right)_{np} := \hat{A}(\mu, n, p) = \sum_{j=1}^{M^a} \alpha_j(\mu) \hat{A}_{j, np}
\]

- matrices \( \{\hat{A}_j\}_{j=1}^{M^a} \) are precomputed offline
- coefficients \( \{\alpha_j(\mu)\}_{j=1}^{M^a} \) are evaluated online
- straightforward for affine transformations

For the matrix \( \hat{K}(\mu, \hat{v}) \), search for a separated approximation

\[
k(\mu, u(\mu); \phi_n)(h(\mu)(\bar{x})) =: \kappa(\mu, n, \bar{x}) \approx \sum_{j=1}^{M^k} \varphi_j(\mu) q_j(n, \bar{x})
\]

The functions \( \{\varphi_j, q_j\}_{j=1}^{M^k} \) are built offline using the EIM

*Barrault, Maday, Nguyen, Patera (’04)*
Offline/Online efficient RB problem

**Offline**: using $\{q_j\}_{j=1}^{M^k}$, we build
- a matrix $B \in \mathbb{R}^{M^k \times M^k}$
- a family of matrices $\{C_j\}_{j=1}^{M^k}$ all in $\mathbb{R}^{R \times N}$

**Online**: For each new parameter $\mu \in \mathcal{P}$ and a vector $\hat{\nu} \in \mathbb{R}^N$
- compute a family of functions $\{\hat{\phi}_j(\mu, \hat{\nu})\}_{j=1}^{M^k}$
- assemble

$$\hat{K}(\mu, \hat{\nu}) \approx D^k(\mu, \hat{\nu}) := \sum_{i,j=1}^{M^k} C_j B_{ji} \hat{K}_i(\mu, \hat{\nu}) \in \mathbb{R}^{R \times N}$$

Similarly, we build an efficient approximation $\hat{\gamma}(\mu, \hat{\nu})$ for the gap function
Offline/Online efficient RB problem

Solve

\[
(\hat{u}(\mu), \hat{\lambda}(\mu)) = \arg \min_{\hat{\nu} \in \mathbb{R}^N, \hat{\eta} \in \mathbb{R}^R} \max \left\{ \frac{1}{2} \hat{\nu}^T \hat{A}(\mu) \hat{\nu} - \hat{\nu}^T \hat{f}(\mu) \right. \\
+ \left. \hat{\eta}^T (D^\kappa(\mu, \hat{\nu}) \hat{\nu} - D^\gamma \hat{\gamma}(\mu, \hat{\nu})) \right\}
\]

\[
D^\kappa(\mu, \hat{\nu}) \in \mathbb{R}^{R \times N} \quad D^\gamma \in \mathbb{R}^{R \times M^\gamma} \quad \hat{\gamma}(\mu, \hat{\nu}) \in \mathbb{R}^{M^\gamma}
\]

- \(D^\kappa(\mu, \hat{\nu})\) results from the EIM on \(\kappa\)
- \(D^\gamma\) results from the EIM on \(\gamma\)
Online stage

Algorithm 1 Online stage

Input: \( \mu \)

\[
\left\{ \hat{f}_j \right\}_{1 \leq j \leq J_f}, \left\{ \hat{A}_j \right\}_{1 \leq j \leq J_a} \\
\left\{ (n^K_i, x^K_i) \right\}_{1 \leq i \leq M^k}, \left\{ q^K_j \right\}_{1 \leq j \leq M^k} \\
\left\{ x^\gamma_i \right\}_{1 \leq i \leq M^g}, \left\{ q^\gamma_j \right\}_{1 \leq j \leq M^g}, B^K, \left\{ C^K_j \right\}_{1 \leq j \leq M^k} \text{ and } D^\gamma
\]

1: Assemble the vector \( \hat{f}(\mu) \) and the matrix \( \hat{A}(\mu) \)
2: Compute \( \hat{\kappa}(\mu, \hat{v}) \) and \( \hat{\gamma}(\mu, \hat{v}) \)  
   EIM on the contact map
3: Compute \( D^K(\mu) \) using \( \hat{\kappa}(\mu, \hat{v}) \)  
   EIM on the gap map
4: Solve the reduced saddle-point problem to obtain \( \hat{u}(\mu) \) and \( \hat{\lambda}(\mu) \)

Output: \( \hat{u}(\mu) \) and \( \hat{\lambda}(\mu) \)
Basis constructions

Two goals

1. Build $\tilde{V}_N \subset \tilde{V}_N$ of dimension $N \ll N$  
   ⇒ POD

2. Build $\tilde{W}_R \subset \tilde{W}_R$ of dimension $R \ll R$
   Requirement: Positive basis vectors
Basis constructions

Two goals

1. Build $\tilde{V}_N \subset \tilde{V}_N$ of dimension $N \ll N$  
   $\Rightarrow$ POD✓

2. Build $\tilde{W}_R \subset \tilde{W}_R$ of dimension $R \ll R$  
   Requirement: Positive basis vectors  
   - POD✓
   - NMF✓: considered in Balajewicz, Amsallem, Farhat (’16)
   - Angle-greedy algorithm✓: introduced in Haasdonk, Salomon, Wohlmuth (’12)
   - Cone-projected greedy algorithm✓✓: devised in this work (19’)

Amina Benaceur  A RBM for variational inequalities  16/29
Non-negative Matrix Factorization

We search for $W$

$$W = \text{NMF}(T, R)$$

- Input: integer $R$
- Output: $R$ positive vectors

*Lee, Seung ('01)*
Non-negative Matrix Factorization

We search for \( \mathbf{W} \)

\[
\mathbf{W} = \text{NMF}(\mathbf{T}, \mathbf{R})
\]

- Input: integer \( \mathbf{R} \)
- Output: \( \mathbf{R} \) positive vectors

Lee, Seung ('01)

NMF optimization problem

\[
(\mathbf{W}, \mathbf{H}) = \arg\min_{\mathbf{W} \in \mathbb{R}^{R \times R}_+, \mathbf{H} \in \mathbb{R}^{R \times P}_+} \| \mathbf{T} - \tilde{\mathbf{W}}\tilde{\mathbf{H}} \|^2
\]

Functional \( \| \mathbf{T} - \tilde{\mathbf{W}}\tilde{\mathbf{H}} \| \) is not convex in both variables \( \tilde{\mathbf{W}} \) and \( \tilde{\mathbf{H}} \) together \( \implies \) only local minima

- Non-unique solution
Non-negative Matrix Factorization

We search for $W$

$$W = \text{NMF}(T, R)$$

- Input: integer $R$
- Output: $R$ positive vectors

Lee, Seung ('01)

NMF optimization problem

$$(W, H) = \underset{T \in R_{+} \times P, Z \in R_{+} \times R}{\text{argmin}} \| T - \tilde{W}ZZ^{-1}\tilde{H} \|^2$$

Functional $\| T - \tilde{W}\tilde{H} \|$ is not convex in both variables $\tilde{W}$ and $\tilde{H}$ together $\implies$ only local minima
- Non-unique solution
Non-negative Matrix Factorization

We search for $W$

$$W = \text{NMF}(T, R)$$

- Input: integer $R$

- Output: $R$ positive vectors

Lee, Seung ('01)

NMF optimization problem

$$(W, H) = \arg\min_{W \in \mathbb{R}_+^{R \times R}, \tilde{H} \in \mathbb{R}_+^{R \times P}} \|T - \tilde{W}ZZ^{-1}\tilde{H}\|^2$$

Functional $\|T - \tilde{W}\tilde{H}\|$ is not convex in both variables $\tilde{W}$ and $\tilde{H}$ together $\implies$ only local minima

- Non-unique solution 😞
Angle-greedy algorithm

### Selection criterion

\[ \mu_n \in \arg\max_{\mu \in P^{tr}} \| z \left( \tilde{\lambda}(\mu; \cdot), \tilde{W}_{n-1} \right) \|_{\ell^\infty(\tilde{T}_1^{c, tr})} \]

### Dual set at iteration \( n \)

\[ \tilde{K}_n := \text{span}_+ \{ \tilde{\lambda}(\mu_1; \cdot), \ldots, \tilde{\lambda}(\mu_n; \cdot) \} \]

### Stopping criterion

\[ r_n < \epsilon_{du} \]

- **Input:** Tolerance \( \epsilon_{du} \) to reach
- **Output:** \( R \) positive vectors
- \( \tilde{W}_n \) is a linear space
Angle-greedy algorithm

Selection criterion

\[ \mu_n \in \arg\max_{\mu \in P^{tr}} \| \tilde{\lambda}(\mu; \cdot) - \Pi_{\tilde{W}_{n-1}} (\tilde{\lambda}(\mu; \cdot)) \|_{\ell^\infty(\tilde{T}^{c, tr})} \]

Dual set at iteration \( n \)

\[ \tilde{K}_n := \text{span}_+ \{ \tilde{\lambda}(\mu_1; \cdot), \ldots, \tilde{\lambda}(\mu_n; \cdot) \} \]

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Angle-greedy algorithm

Selection criterion

\[ \mu_n \in \arg\max_{\mu \in \mathcal{P}^\text{tr}} \| \tilde{\lambda}(\mu; \cdot) - \Pi_{\tilde{W}_{n-1}}(\tilde{\lambda}(\mu; \cdot)) \|_{\ell^\infty(\tilde{\Gamma}_{1,\text{tr}})} \]

Dual set at iteration \( n \)

\[ \tilde{K}_n := \text{span}_+ \{ \tilde{\lambda}(\mu_1; \cdot), \ldots, \tilde{\lambda}(\mu_n; \cdot) \} \]

Stopping criterion

\[ r_n < \epsilon_{\text{du}} \]

- Input: Tolerance \( \epsilon_{\text{du}} \) to reach 😞
- Output: \( R \) positive vectors
- \( \tilde{W}_n \) is a linear space 😞
Cone-projected greedy algorithm

**Selection criterion**

\[ \mu_n \in \arg\max_{\mu \in \mathcal{P}^{\text{tr}}} \| \breve{\lambda}(\mu; \cdot) - \Pi_{\mathcal{K}_{n-1}}(\breve{\lambda}(\mu; \cdot)) \|_{\infty(\mathcal{T}^c_{1,\text{tr}})} \]

**Dual set at iteration** \( n \)

\[ \tilde{\mathcal{K}}_n := \text{span}_+\{ \tilde{\mathcal{K}}_{n-1}, \tilde{\lambda}(\mu_n; \cdot) \} \]

**Stopping criterion**

\[ r_n < \epsilon_{\text{du}} \]

→ Python cvxopt library for positive projections

- Input: Tolerance \( \epsilon_{\text{du}} \) to reach 😊
- Output: \( R \) positive vectors
Cone-projected greedy algorithm

Algorithm 2 Cone-projected weak greedy algorithm

Input : $P^\text{tr}$, $\tilde{\Gamma}_1^c$ and $\epsilon_{\text{du}} > 0$

1: Compute $S_{\text{du}} = \{\tilde{\lambda}(\mu; \cdot)\}_{\mu \in P^\text{tr}}$

2: Set $\tilde{K}_0 = \{0\}$, $n = 1$ and $r_1 = \max_{\mu \in P^\text{tr}} \|\tilde{\lambda}(\mu; \cdot)\|_{\ell^\infty(\tilde{\Gamma}_1^c)}$

3: while ($r_n > \epsilon_{\text{du}}$) do

4: Search $\mu_n \in \arg\max_{\mu \in P^\text{tr}} \|\tilde{\lambda}(\mu; \cdot) - \Pi_{\tilde{K}_{n-1}}(\tilde{\lambda}(\mu; \cdot))\|_{\ell^\infty(\tilde{\Gamma}_1^c, \text{tr})}$

5: Set $\tilde{K}_n := \text{span}_+\{\tilde{K}_{n-1}, \tilde{\lambda}(\mu_n; \cdot)\}$

6: Set $n = n + 1$

7: Set $r_n := \max_{\mu \in P^\text{tr}} \|\tilde{\lambda}(\mu; \cdot) - \Pi_{\tilde{K}_{n-1}}(\tilde{\lambda}(\mu; \cdot))\|_{\ell^\infty(\tilde{\Gamma}_1^c)}$

8: end while

9: Set $R := n - 1$

Output : $\tilde{W}_R := \tilde{K}_R$

→ Python cvxopt library for positive projections
**Hertz's half-disks problem**

- Imposed displacement on the upper half-sphere
- Parametric radius $\mu$ for the lower half-sphere
- $\mathcal{P} = [0.9, 1.12]$, $\mathcal{P}^{tr} = \{0.905 + 0.01i | 0 \leq i \leq 22\}$
- Non-matching meshes, $\mathcal{N} = 1350$ nodes, $\mathcal{R} = 51$
High-fidelity solution procedure

- Conforming FEM for displacement, $dG \mathbb{P}_0$ for contact pressures
- Collocation at contact nodes (possible instability)
- Local Averaged Contact (LAC): macro-elements (two consecutive segments) (Drouet, Hild (’17))

$\mu = 0.9$ (left) / 1.12 (right), coarse (top) / fine (bottom) meshes
Dual basis dimension $R$ as a function of the truncation threshold $\epsilon_{du}$

<table>
<thead>
<tr>
<th>$\epsilon_{du}$</th>
<th>$5 \cdot 10^{-2}$</th>
<th>$10^{-2}$</th>
<th>$5 \cdot 10^{-3}$</th>
<th>$10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMF $R$</td>
<td>4</td>
<td>10</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>Cone-projected greedy $R$</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>13</td>
</tr>
</tbody>
</table>
Empirical interpolation

EIM error as a function of the rank $M$ of the EIM approximation.
Error estimation

Error on the minimum energy $e_{\text{ener}}(\mu)$

Relative $H^1$-error for the displacement field $e_{\text{displ}}(\mu)$
Constraint violation

Maximum interpenetration

CPG/NMF(18)

CPG/NMF(5)
Ring on block

- Imposed displacement at the ring’s top ends
- Parametric ring’s radius $\mu \in \mathcal{P} := [0.95, 1.15]$, $\text{card}(\mathcal{P}^{\text{tr}}) = 21$
- Non-matching meshes, $\mathcal{N} = 2 \times 1590$ dofs, $\mathcal{R} = 50$ dofs
- Reference (left) and deformed (right) configuration
Basis construction

Primal basis

Dual basis
Conclusions and perspectives

Contributions

✓ RBM in a general setting
  • varying normals
  • non-matching meshes
✓ Cone-projected greedy algorithm
  • outperforms the NMF
  • outperforms the angle-greedy

Perspectives

? Extensions to other types of contact
  • friction
  • multibody
  • dynamic

? Convergence rates of the CPG

A reduced basis method for parametrized variational inequalities applied to contact mechanics. AB, V. Ehrlacher and A. Ern. IJNME, 2019.